Inflation as a Probe of Short Distance Physics

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We show that a string-inspired Planck scale modification of general relativity can have observable cosmological effects. Specifically, we present a complete analysis of the inflationary perturbation spectrum produced by a phenomenological Lagrangian that has a standard form on large scales but incorporates a string-inspired short distance cutoff, and find a deviation from the standard result. We use the de Sitter calculation as the basis of a qualitative analysis of other inflationary backgrounds, arguing that in these cases the cutoff could have a more pronounced effect, changing the shape of the spectrum. Moreover, the computational approach developed here can be used to provide unambiguous calculations of the perturbation spectrum in other heuristic models that modify trans-Planckian physics and thereby determine their impact on the inflationary perturbation spectrum. Finally, we argue that this model may provide an exception to constraints, recently proposed by Tanaka and Starobinsky, on the ability of Planck-scale physics to modify the cosmological spectrum.

I. INTRODUCTION

In the last two decades, particle physics has become an indispensable part of cosmology. In fact, one of the strong motivations for studying particle physics theories that go beyond the standard model and incorporate gravity is that they may shed light on the nature of the cosmological singularities arising in general relativity. There is widespread hope that in one form or another these and other cosmological considerations may one day allow us to test physical theories whose fundamental scales are now, and perhaps forever, beyond the reach of conventional accelerator experiments.

Inflationary cosmological models, in particular, significantly highlight the important role of microphysics. For example, since the pioneering work of [1] it has been known that quantum field fluctuations in the early universe are stretched by inflationary expansion to scales of astrophysical relevance, providing a gratifying first principles mechanism for structure formation. Galaxies, from this viewpoint, are quantum fluctuations writ large.

In order to solve the standard cosmological puzzles, a minimum of 60 e-folds of inflationary expansion must be invoked, but in many models this number can be *much* larger. Taking this at face value, it means that today's scales of cosmological relevance expanded from Planckian or sub-Planckian scales at the onset of inflation. Inflation may therefore provide a kind of Planck scale microscope, stretching the smallest of distance scales to observably large size.

It is possible, however, that in the process of such inflationary expansion the effect of Planck scale physics gets washed out, being diluted by the very growth of scales which potentially makes them visible. A similar phenomenon has been observed in black hole physics. If one traces the history of a Hawking radiated photon, one finds that it gets ever more blue-shifted toward the moment of its emission, and hence one might think it could carry an imprint of extremely high energy physical processes. In reality though, a number of studies [4] have concluded that such short distance physics does not have any impact on the low energy features of Hawking radiation; heuristically, short distance modifications are washed out by the memoryless process of thermalization.

In a cosmological context, the situation in this regard has been less clear. Brandenberger and Martin [2] carried out a study of the impact of various models of trans-Planckian physics (in fact, the same models considered in the black hole studies just mentioned) on the spectrum of density perturbations in power law inflationary models. They found that while some hypothesized Planck scale modifications to ordinary field theory yield no late time consequences (similar to the black hole conclusion) some do, indicating a cosmological sensitivity to short distance physics. On the other hand, it was argued in [3] that there should be no change in the perturbation spectrum if the proposed modifications still yield the adiabatic vacuum*.

In this paper we take up the issue of cosmological sensitivity to short distance physics, but from a different approach. Namely, rather than considering the *ad hoc* short scale modifications studied in [2,3], we focus on an

*Note that the definitions of adiabaticity used in [2] and [3] are slightly different, leaving the situation somewhat ambiguous.

interesting model introduced in [6], whose short distance physics is designed to incorporate a minimum length, a development naturally inspired by string theory. We present a full analysis of the perturbation spectrum produced by the presence of such a minimum length in a de Sitter background. As with *any* calculation of the inflationary perturbation spectrum, the modes are normalized well inside the horizon, and the calculational task is to track their evolution until they are far outside the horizon, as the perturbation spectrum is fully determined by their asymptotic values. We solve the mode equation numerically to avoid invoking any new analytical approximations, and fix the initial conditions for the numerical solution by matching to an approximate analytical solution of the mode equations that is valid at very early times.

While completing this work, Kempf and Niemeyer [7] posted a complementary calculation of the spectrum in the presence of a short distance cutoff. We contrast our approach with theirs, showing that the numerical evaluation of the mode functions leads to a complete description of the spectrum. We also expect that the approach used here will generalize to the broader problem of determining the impact of trans-Planckian physics on the inflationary perturbation spectrum, which can manifest itself as a change to the dispersion relation [2], or the initial normalization of the modes or choice of vacuum.

In what follows, we briefly review the construction of a model with a minimum length, and discuss the analytical properties of the perturbation equation. In particular, following [6] we show that since there is a minimum length, each mode is "created" at a finite time in the past, defined by the moment when its physical wavelength first exceeds the minimum length. We then show how to extract an approximate analytical solution that holds near this early time. This allows us to "match" this solution to a numerical solution of the full mode equation, and compute the amplitude of the perturbation spectrum as a function of the minimum length. We find that, indeed, there is an imprint of the short distance cutoff on the perturbation spectrum. We also confirm that the scenario we study does not violate bounds coming from late time particle production and quantum mechanical self consistency. More generally, we emphasize that the effect of any short distance modification can be encoded in two wavelength dependent functions, providing a phenomenological approach for systematically analysing deviations from the standard prediction of the power spectrum. We conclude with a brief discussion of the spectrum we might expect in a more general spacetime, and a summary of our conclusions.

We stress at the outset that our intent is to show explicitly that a modification to conventional physics at the Planck scale can have an observable effect on precision cosmological measurements. While of interest in its own right, the model [6] on which we focus should be viewed as one concrete example in which such calculations can be reliably performed, allowing us to establish definitively that cosmological observations may be a window onto Planck scale physics.

II. PERTURBATIONS WITH A MINIMUM LENGTH

Motivated by the stringy uncertainty principle [5], a phenomenological Lagrangian which incorporates a short distance cutoff was recently proposed by Kempf [6]. In this approach, the short distance cutoff is modeled by modifying the usual commutation relation to

$$[\mathbf{x},\mathbf{p}] = i\hbar\left(1+\beta\mathbf{p}^2\right). \qquad (1)$$

The parameter β is related to the minimum distance Δx_{min} by $\Delta x_{min} \sim \sqrt{\beta}$. A Lagrangian suggested by Eq.(1) was discussed in [6]. In this model, the tensor mode $u_{\tilde{k}}$ obeys the following equation of motion:

$$u''_{\tilde{k}} + \frac{\nu'}{\nu}u'_{\tilde{k}} + \left(\mu - \frac{a''}{a} - \frac{a'}{a}\frac{\nu'}{\nu}\right)u_{\tilde{k}} = 0, \qquad (2)$$

where a denotes the scale factor and the prime denotes differentiation with respect to conformal time η. Our $u_{\tilde{k}}$ is equal to $a^2\phi_{\tilde{k}}$ from [6], while $\tilde{k}^i = a\rho^i e^{-\beta\rho^2/2}$ where ρ^i is the Fourier transform of the physical coordinates x^i, and

$$\mu(\eta,\rho) \equiv \frac{a^2\rho^2}{(1-\beta\rho^2)^2}, \qquad \nu(\eta,\rho) \equiv \frac{e^{\frac{3}{2}\beta\rho^2}}{(1-\beta\rho^2)}. \qquad (3)$$

The cutoff is defined by requiring that $\rho^2 \leq 1/\beta$, and is motivated by the minimum distance in string theory. Tensor perturbations with different comoving wavenumber k reach the cutoff $\rho^2 = 1/\beta$ at different conformal time η_k, where $a^2(\eta_k) = \beta k^2$. For de Sitter space, $\eta_k = -1/\sqrt{e\beta H^2\tilde{k}^2}$.

Note that in evaluating the derivatives of $u_{\tilde{k}}$ with respect to η, we are holding $\tilde{k}$ (and not the usual comoving momentum k) fixed with time. It is therefore convenient to express μ and ν in terms of $\tilde{k}$ by introducing the product-log, or Lambert W function [9], which is defined so that $W(xe^x) = x$:

$$\mu = -\frac{a^2}{\beta}\frac{W(z)}{(1+W(z))^2}\ , \quad \frac{\nu'}{\nu} = \frac{a'}{a}\frac{W(z)\,(5+3W(z))}{(1+W(z))^2}\ . \qquad (4)$$

where $z = -\beta\tilde{k}^2/a^2$. The W function has an essential singularity when its argument is equal to $-1/e$, and this corresponds to the precise moment (for a given k) when $\eta = \eta_k$. Let us examine in more detail the nature of the singularity in the equation of motion at $\eta = \eta_k$. The

Lambert W function $W(z)$ has a series expansion near the branch point $z = -1/e$ [9]:

$$W(z) = -1 + p - \frac{1}{3}p^2 + \frac{11}{72}p^3 + \dots \tag{5}$$

where $p = \sqrt{2(ez+1)}$. The series converges for $|p| < \sqrt{2}$. The singular point at $\eta = \eta_k$ is irregular because the coefficients of $u'_{\tilde{k}}$ and $u_{\tilde{k}}$ are not analytic in $\eta - \eta_k$. However, the non-analytic piece is less singular than $1/(\eta - \eta_k)$ and is therefore subdominant. We can solve for the leading behavior of $u_{\tilde{k}}$ by extracting the most singular terms of the equation of motion. First, write $\eta = \eta_k(1-y)$, so that the leading terms in the equation of motion give

$$\ddot{u}_{\tilde{k}} - \frac{1}{2y}\dot{u}_{\tilde{k}} + \left(\frac{1}{2\beta H^2} + 1\right)\frac{1}{2y}u_{\tilde{k}} \tag{6}$$

where dot denotes derivatives with respect to y. Written in this form, the $\dot{u}_{\tilde{k}}$ and $u_{\tilde{k}}$ terms both have divergences that scale as $1/(\eta - \eta_k)$, so the equation for the leading behavior of $u_{\tilde{k}}$ actually has a *regular* singular point at η_k.

Since any second order differential equation has a power-law expansion about a regular singular point [10], we can explicitly construct the two independent solutions of Eq.(6) in the vicinity of η_k. Let $u_{\tilde{k}}$ take the form:

$$u_{\tilde{k}} = y^\alpha \sum_{n=0}^{\infty} c_n y^n. \tag{7}$$

The *indicial equation* is obtained by inserting this expression into into Eq.(6), expanding about $y = 0$ and indentifying the lowest order terms, which are proportional to $y^{\alpha-2}$. This gives $\alpha = 0$ or $3/2$. Therefore, for small y,

$$u_{\tilde{k}} \sim c_1 + c_2 y^{3/2}. \tag{8}$$

The coefficients $c_{1,2}$ are constrained by the Wronskian condition which follows from the canonical commutation relation $\left[\phi_{\tilde{k}}, \pi_{\tilde{r}}\right] = i\delta^3(\tilde{k} - \tilde{r})$ for $\phi_{\tilde{k}} = u_{\tilde{k}}/a$ and its conjugate momentum $\pi_{\tilde{r}}$ [6,7] [†]:

$$u_{\tilde{k}}(\eta)u^{*\prime}_{\tilde{k}}(\eta) - u^*_{\tilde{k}}(\eta)u'_{\tilde{k}}(\eta) = i\left(1 - \beta\rho^2\right)\exp(-\frac{3}{2}\beta\rho^2). \tag{9}$$

Hence,

$$c_1 c_2^* - c_1^* c_2 = i\eta_k \frac{4}{3} e^{-3/2}. \tag{10}$$

The Wronskian condition mixes the two power series solutions with complex coefficients. *A priori*, there is a one parameter family of solutions of $c_{1,2}$, corresponding to different choices of the vacuum state. In the usual inflationary scenario, the natural choice of the vacuum state is the so-called Bunch-Davies vacuum which reduces to the Minkowskian vacuum for wavelengths much shorter than the Hubble scale. We now motivate a natural choice of vacuum in the present context by comparing with the Bunch-Davies vacuum. Recall that in the standard setting, the high frequency limit of the mode equation takes the form

$$u''_k(\eta) + \omega_k^2(\eta)u_k(\eta) = 0, \tag{11}$$

whose solution is well approximated by the WKB form

$$u^{\mp}_{\tilde{k}}(\eta) = \frac{1}{\sqrt{2\omega_k}}\exp(\pm i\int^\eta \omega_k(\eta')d\eta') \tag{12}$$

if the adiabatic conditions $\frac{\omega''_k}{\omega_k} \ll 1$ and $\left|\frac{\omega'_k}{\omega_k^2}\right| \ll 1$ are both satisfied. The Bunch-Davies vacuum amounts to choosing the Heisenberg form of the field operator $\hat{u}(\eta, \mathbf{x})$ to be

$$\hat{u}(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}(u^+_k a_k e^{ikx} + u^{*+}_k a^\dagger_k e^{-ikx}), \tag{13}$$

with a vacuum state $|0\rangle$ satisfying $a_k|0\rangle = 0$. In our case, if we could ignore the $\dot{u}_{\tilde{k}}$ in (6), similar reasoning suggests

$$u^+_{\tilde{k}}(\eta) = \frac{1}{\sqrt{2\omega_k}}\exp(-i\int^\eta \omega_k(\eta')d\eta'), \tag{14}$$

where, from (6), we see that for $\eta \sim \eta_k$,

$$\omega_k^2 = \frac{A}{\eta_k^2 y} \tag{15}$$

and

$$A = \frac{1}{4\beta H^2} + \frac{1}{2}. \tag{16}$$

This would give a Bunch-Davies-like vacuum of the form

$$u^+_{\tilde{k}}(y) = \left(\frac{\eta_k^2 y}{4A}\right)^{1/4}\exp(-2i\sqrt{Ay}). \tag{17}$$

Of course, though, we can not ignore the $\dot{u}_{\tilde{k}}$ in (6), and hence this does not yield a solution to our modified mode equation. However, by modifying the prefactor of the exponential, we can in fact construct a solution of this form,

$$F(y) = \left(\frac{\sqrt{A}}{2} + iA\sqrt{y}\right)\exp(-2i\sqrt{Ay}). \tag{18}$$

[†]In the model of [6], the commutation relations of $\mathbf{x}$ and $\mathbf{p}$ are modified as in Eq.(1), but the field commutation relations are not changed from their standard form. While one can seek a particular interpretation, here we simply view this as providing an interesting, short distance modification of the field equations that allows for a reliable calculation of cosmological implications.

This solves Eq. (6); equivalently, in the series expansion of $F(y)$ the $y^{1/2}$ term cancels out, and $F(y)$ is seen to be a linear combination of the two power series solutions found above with $c_1 = \frac{\sqrt{A}}{2}$ and $c_2 = -i\frac{4}{3}A^2$.

The general solution is a linear combination of the positive and the negative frequency modes

$$u_{\tilde{k}}(y) = C_+ F(y) + C_- F^*(y) \tag{19}$$

with the constants $C_\pm$ constrained by the Wronskian condition

$$|C_+|^2 - |C_-|^2 = \frac{e^{-2}}{\sqrt{\beta}\tilde{k}H}\left(\frac{1}{4\beta H^2} + \frac{1}{2}\right)^{-5/2}. \tag{20}$$

Comparison of the osciallatory part of $F(y)$ with that of the Bunch-Davies vacuum suggests that $C_- = 0$. However, the normalization of $F(y)$ is not $1/\sqrt{2\omega}$ since the adiabatic condition is not satisfied. To be specific,

$$\frac{\omega_k'}{\omega_k^2} = \frac{1}{2\sqrt{Ay}}. \tag{21}$$

Therefore for $y \sim 0$, no matter how large A is (or, more importantly, how small β is), the adiabatic condition $|\frac{\omega_k'}{\omega_k^2}| << 1$ is violated. Thus these initial conditions are not smoothly connected to the "standard" form of the mode equation when it is well inside the horizon, even in the limit $\beta \to 0$ although the Lagrangian is smooth in the same limit. The discrepancy arises because whenever $\beta \neq 0$, the mode's evolution begins at a *finite* conformal time. Moreover, we expanded the Lambert function $W(z)$ with $z = -\beta\tilde{k}^2/a^2$ around $z = -1/e$, and this expansion is only convergent when $z < 0$. If $\beta = 0$, the argument of the W function is zero, and this expansion cannot be used.

We have found the leading behavior of $u_{\tilde{k}}$ around $\eta \sim \eta_k$. The equation of motion for the tensor mode $u_{\tilde{k}}$ is solved only up to order $1/y$. The residual terms can still be significant for $\eta \sim \eta_k$. We deduce the subleading bahavior of $u_{\tilde{k}}$ by the method of dominant balance [11]. Define $u_{\tilde{k}}(y) = F(y)\,(1 + \epsilon_1(y))$ and extract the most singular terms in the equation of motion for ϵ_1:

$$\ddot{\epsilon}_1 - \frac{1}{2y}\dot{\epsilon}_1 = \frac{3A - 3/2}{\sqrt{y}} = 0, \tag{22}$$

which gives

$$\epsilon_1(y) = (2A-1)y^{3/2}\left(\log y - \frac{2}{3}\right). \tag{23}$$

This is indeed small compared with the leading term for small y. With this correction, the equation is solved up to order $1/\sqrt{y}$, but there are still residual $\ln(y)$ terms. The solution is further improved by the next subleading order $\epsilon_2(y)$, where $u_{\tilde{k}} = F(y)(1+\epsilon_1(y))(1+\epsilon_2(y))$. The solution for $\epsilon_2(y)$ is

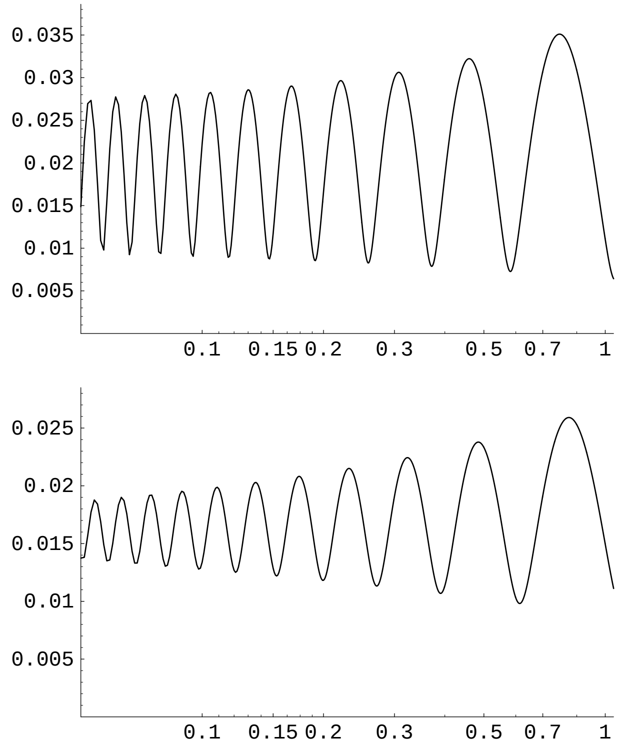

FIG. 1. The top figure shows the spectrum ($P^{1/2}$) as a function of β with $C_-/C_+ = -.5$, while the lower plot shows the spectrum when $C_- = 0$. It is only in the latter case the computed value of the spectrum smoothly approaches the usual value in de Sitter space, In this plot $H = .1$, and with $\beta = 0$ we would expect $P^{1/2} = 0.0159155$.

$$\epsilon_2 = \frac{1}{24}\left(105 - 330A - 112iA^{3/2}\right)y^2 + \frac{7}{4}(2A-1)\,y^2\log y. \tag{24}$$

The equation of motion for $u_{\tilde{k}}$ is therefore solved up to terms that vanish as $\eta \to 0$.

When the mode is well outside the horizon, $\rho \ll H$, and see that $\mu(\eta,\rho) \to k^2$ and $\nu(\eta,\rho) \to 1$. Consequently, $u_k(\eta) \sim a(\eta)$, which reproduces the standard late time limit in the usual case with $\beta = 0$. This limit defines the power spectrum,

$$\mathcal{P}_g(k) = \frac{k^3}{2\pi^2}\left|\frac{u_k}{a}\right|^2. \tag{25}$$

which is evaluated when u_k is well outside the horizon.

III. NUMERICAL RESULTS

We match the analytical form of the solution valid when the mode is well inside the horizon to a numerical evaluation of the full mode equation. In principle we

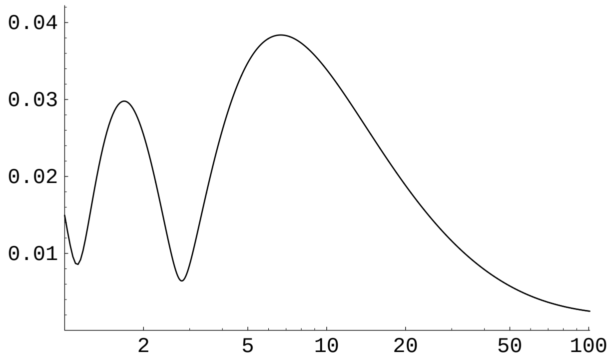

FIG. 2. The β dependence of the spectrum, $P_k^{1/2}$, is plotted against β, with $H = 0.1$. For a de Sitter background, the spectrum is independent of k.

might hope to evaluate the mode equation numerically from an initial time of $y = 0$, but the coefficients of $u_{\tilde{k}}$ and $u'_{\tilde{k}}$ are infinite at this point so it is easier to match the numerical solutions to the approximate analytical solutions at y_0, a small but finite value of y. By varying y_0 we can ensure that our results do not depend on our choice of starting point, and that we are therefore "close enough" to $y = 0$.

A subtle point that arises during the integration is that the subdominant terms in $u_{\tilde{k}}$ contribute terms of order $y^{3/2}\log(y)$, which contribute $\sqrt{y}\log y$ in the derivative, $u'_{\tilde{k}}$. While these terms do go to zero in the limit where y_0 is vanishingly small, they approach zero comparatively slowly. Since we are using a small but not infinitesimal value of y_0, we match the numerical solution to the "corrected" asymptotic solution, which includes the lowest order logarithmic terms. We evolve the mode equations numerically using the Bulrisch-Stoer integrator implemented in Fortran, and from the asymptotic values of $|u_k/a|$ we obtain the spectrum, $P^{1/2}(k)$.

In general the spectrum is k-dependent, but for the special case of de Sitter inflation, the background spacetime is time translation invariant, which implies that the spectrum should not depend on k (which is equivalent to $\tilde{k}$ at late times). This is manifest from the mode equation and, in practice, we verified the code by solving the mode equation for multiple values of k and found that the numerically computed spectrum was indeed scale invariant to better than 1 part in 10^6.

In the previous section, we saw that imposing the Wronskian constraint led to a one parameter family of solutions, but that a comparison with the standard Bunch-Davies vacuum suggests the choice $C_- = 0$. We begin by analysing the consequences of relaxing this choice. In Fig. (1) we plot the spectrum computed for small values of β with $C_- = 0$, and compare this plot to a calculation with a finite value of C_-. In the former case, the spectrum smoothly approaches the de Sitter result $P^{1/2} = H/2\pi$, but does not do so for any finite value of C_-.

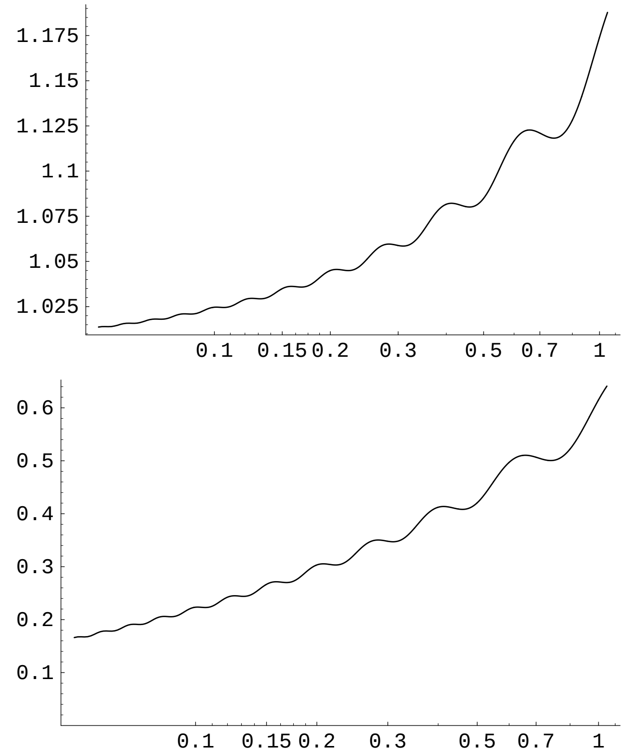

FIG. 3. The values of D_+ (top) and D_- are extracted for the spectrum shown in Fig. 1 with $C_- = 0$. As $\beta \to 0$, $D_+ \to 1$ and $D_- \to 0$, confirming that as the Hubble length becomes much larger than the minimum length the spectrum approaches the standard result.

Fig. 2 displays the β dependence of the spectrum. For moderate values of β, the spectrum can be either lower or higher than the "standard" value, but for large β, $P^{1/2}$ approaches zero. For sufficiently large β, the overall amplitude of the perturbation spectrum is significantly reduced. In de Sitter space the power spectrum is k-independent, so the only effect is an overall normalization dependence on β through the dimensionless combination $\sqrt{\beta}H$. Assuming one has an independent measure of H, we see that the short distance cutoff specified by β does indeed leave an imprint on the spectrum. In more realistic inflationary models, as we indicate later, we believe the imprint will be k-dependent and hence also affect the shape of the power spectrum.

IV. MODE MATCHING AFTER HORIZON CROSSING

The physical origin of the change in normalization of the power spectrum can be simply understood in terms of the asymptotic behavior of the mode solutions. In the long wavelength limit, the mode equation asymptotes to the standard mode equation for cosmological perturbations. Therefore, regardless of the nature of the short-distance physics, the mode function takes the general form:

$$u_k(\eta) = \frac{1}{2}\sqrt{-\pi\eta}\left[D_+ H_\nu(-k\eta) + D_- H_\nu^*(-k\eta)\right], \quad (26)$$

where H_ν is a Hankel function of the first kind, and $v = 3/2$ in de Sitter space. The constants $D_\pm$ are constrained by the standard Wronskian condition arising from the canonical commutation relations for the field operator:

$$u_k^* \frac{du_k}{d\eta} - u_k \frac{du_k^*}{d\eta} = -i. \quad (27)$$

With this constraint, the form of the mode function is completely determined (up to a phase) by the selection of a vacuum. The standard case, that of a Bunch-Davies vacuum at short wavelength, corresponds to the selection $D_- = 0$ and $D_+ = 1$, with our choice of normalization.

The key attribute of this construction is that *regardless* of the nature of the short-distance physics, the long-wavelength behavior of the modes is completely encoded in the coefficients $D_\pm$. In general the $D_\pm$ are k-dependent but in de Sitter space the situation is particularly simple, since the $D_\pm$ are constants.

In the standard analysis, the coefficients $C_\pm$ that determine the vacuum at short wavelengths are identical to the $D_\pm$ found by matching the mode functions to the long-wavelength limit, Eq. (26). Modifying the short distance physics can break this correspondence, and the choice of vacuum $C_- = 0$ in the short wavelength limit can correspond to a phase rotation in the long wavelength limit, with $D_- \neq 0$. This changes the normalization of u_k, since the power spectrum depends on the integration constants as

$$\mathcal{P}_g(k) = \frac{k^3}{2\pi^2}\left|\frac{u_k}{a}\right|^2 \propto |D_+ + D_-|^2 \quad (28)$$

in the long wavelength ($k \to 0$) limit. In de Sitter space, the $D_\pm$ are constant, so the rotation alters the normalization of the power spectrum, but does not introduce any k dependence. However, if the background deviates from de Sitter the $D_\pm$ are, in general, k-dependent, which will alter both the shape and normalization of the power spectrum as well as the short distance physics – perhaps in a dramatic way [8]. Naturally, if a modification to short-distance physics alters $D_\pm$, the perturbation spectrum will change. This will lead to constraints on the form and magnitude of the modifications to $D_\pm$.

V. PARTICLE PRODUCTION BOUNDS

In addition to the perturbation spectrum, other arguments place cosmological constraints on modifications to short-distance physics. For example, Starobinsky [14] has recently shown on quite general grounds that even a very tiny rotation away from the Bunch-Davies vacuum in *the current universe* would result in unacceptable production of relativistic particles. The Starobinsky bound corresponds to

$$|D_-|^2 \leq \frac{H_0^2}{M_{\rm Pl}^2}, \quad (29)$$

where the coefficient D_- is evaluated in the limit of a scale much larger than the minimum distance $\sqrt{\beta}$ but still much smaller than the current horizon size H_0^{-1}. However, this bound applies to the cosmic vacuum at late times, not during inflation. If $|D_-|$ is dependent on H, it can be significant during inflation, thus affecting the power spectrum, but small enough universe today to satisfy the Starobinsky bound.

As it happens, an H dependent value of D_- is precisely what the model considered in this paper predicts. The magnitude of the rotation depends not on the absolute physical scale of the cuttoff, but on the ratio of the cutoff scale to the horizon size, $\sqrt{\beta}H$. If the horizon size and the minimum length are comparable during inflation, the power spectrum is significantly affected, but the rate of particle production today will be strongly suppressed, by the much lower value of H in the present universe. Fig. 3 shows the dependence of the coefficient D_- on the ratio βH^2 in the numerical solution of the mode equation. The numerical results imply $|D_-| \propto \beta^n$, with $0.45 \lesssim n \leq 0.5$ when $\beta H^2 \ll 1$. In the "real" inflationary universe, we hold β fixed and reduce H, so this result implies that as the universe expands the D_+ term will become negligible.

On the basis of our present calculations, the exact dependence of D_- on β remains unclear, since the value of n appears to be very weakly dependent on β: if we evaluate n over a few decades in β (for fixed H) n appears to slowly approach 0.5 as this range is moved to smaller and smaller β. Thus, it is not unreasonable to assume that $|D_-| \propto \beta^{.5}$ in the present universe, and that the particle production rate today is consistent with the Starobinsky bound.

We do not pursue this in more detail as the current model is simply a heuristic construction that mimics what we might expect from a more rigorous string theoretic description of spacetime. However, we can conclude that it is plausible that a minimum length leads to a time dependent modification of the backgroun in a universe where H is not constant, and that the value of D_- will be time dependent making the particule production bound much less onerous.

Also of relevance is Tanaka's analysis [15], showing that particle production during inflation can significantly change the background evolution of the spacetime. The perturbation to the stress-energy during inflation can be expressed as:

$$\begin{aligned}\delta T^{00} &\sim \int \frac{d^3k}{(2\pi)^3}\left[\frac{k\left|D_-\right|^2}{a^4}\right] \\ &= \int \frac{d^3p}{(2\pi)^3} p\left|D_-\right|^2 \\ &\sim \frac{\left|D_-\right|^2}{\beta^2},\end{aligned} \tag{30}$$

where p is the physical momentum $p \equiv (k/a)$, with the momentum cutoff $p < \left(1/\sqrt{\beta}\right)$. If perturbation theory is to be consistent, the contribution from the stress-energy from particle production must be subdominant,

$$\delta T^{00} \sim \frac{\left|D_-\right|^2}{\beta^2} << M_{\rm Pl}^2 H^2. \tag{31}$$

Tanaka's analysis assumed $|D_-| \sim 1$, resulting in a bound on β of

$$\frac{1}{\beta} << \sqrt{M_{\rm Pl} H^2}. \tag{32}$$

In this model, D_- is β-dependent, weakening of the Tanaka bound, and for $D_- \sim \sqrt{\beta} H$ it becomes

$$\frac{1}{\beta} << M_{\rm Pl}^2. \tag{33}$$

This is easily satisfied if $\sqrt{\beta}$ is within an order of magnitude or two of the Planck length. Therefore particle production places no significant constraints on the viability of this model.

VI. DISCUSSION

We have presented an accurate calculation of the perturbation spectrum predicted for a de Sitter universe where the physics has been modified to include a minimum length. The spectrum is scale independent, which follows from the time translation invariance of a de Sitter universe. This calculation significantly extends the contemporaneous qualitative approach of [7], allowing us to determine *how* the perturbation spectrum is modified. Furthermore, it lays the groundwork for accurate calculations of the spectrum produced by other models where trans-Planckian physics has modified the "standard" evolution of cosmological perturbations, either by altering the dispersion relationship in the evolution equations, or by changing the initial conditions.

If the minimum length is much smaller than the Hubble length, its introduction has no detectable effect on the spectrum. However if these two lengths are within an order of magnitude or two, it is possible for the resulting spectrum to differ appreciably from the $\beta = 0$ limit and for various other particle production constraints, discussed above, to be satisfied. Various values for the string scale and the Hubble scale arise in recent string theoretic approaches to cosmology (see for example [16–22]) and hence the effects studied here can potentially be significant in these and other models.

In de Sitter space, the ratio between the minimum length and the physical horizon size is constant. In almost all other inflationary backgrounds, the expansion rate is slower than exponential, and the physical horizon size will increase relative to the minimum length scale. In this case our analysis of the de Sitter background suggests that the amplitude of the longest modes (produced earliest in inflation) will be modified. Shorter modes will leave the horizon at a time when the horizon length is much larger than the physical cut-off length and their amplitude will be unaffected. We plan to return to this problem in future work, but our tentative conclusion is that the spectrum of primordial perturbations could be altered at very long wavelengths by the existence of a minimum length scale. Whether this is observable in practice will depend crucially on the number of e-folds of inflation preceding the creation of the modes which are responsible for large scale structure in the observable universe.

We close with a few observations, some rather speculative, about further studies we plan to undertake based on the present work [8].

• All of the calculations in this paper have focused on tensor perturbations. It would be worthwhile to extend the analysis to scalar perturbations.

• We find it particularly interesting that since short scale modifications yield mode equations that are asymptotic to the standard form on large scales (that is what is meant by a short scale modification) the effect on the power spectrum of *any* new short distance physics can be encoded in the k-dependent coefficients D_+ and D_-. This provides a phenomenological framework for systematically seeking signals of – and establishing constraints on – short scale deviations from conventional physics.

• For very large β, the power spectrum becomes arbitrarily small. It is tempting to interpret this as a mechanism for solving the fine tuning problem endemic to the inflationary generation of the primordial perturbation spectrum. A mechanism ensuring that the overall normalization of the spectrum is small enough to satisfy observational constraints from Large Scale Structure and the microwave background will solve this problem. For large β, this model appears to do just that, but we caution that this conclusion is likely naïve since it requires the minimum physical length to be much larger than the horizon volume, and we can not be sure the model has any physical validity in this regime. Further study along these lines may reveal a trustworthy suppression mechanism.

• An interesting – but highly speculative – possibility is that particle production induced by a finite value of D_- may both be small enough to satisfy Starobinsky's bound, but large enough to lead to new physics. Epochs where this would be particularly intriguing are immediately after inflation, where particle production from the vacuum is a potential mechanism for reheating the universe, and in the present universe if the particle production was efficient enough to modify the equation of state. To explain the latter possibility, the equation of state determines the relationship between the density and the scale factor as the universe expands and particle production reduces the rate at which the density of the universe drops with increasing volume. In general, the more weakly the density of a perfect fluid depends on volume, the more rapidly the universe will expand. Consequently, if particle production is efficient enough to alter the expansion rate of the universe, it will be increased relative to that of a universe with $D_- = 0$. This is particularly interesting in the light of observational evidence for dark energy, which is betrayed by a too-rapid expansion of the spacetime background. For an alternative approach of understanding the origin of dark energy from trans-Planckian physics, see [23].

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